# Homotopietheorie Seminar: Simplicial homotopy theory

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Simplicial sets offer a combinatorial way to deal with the homotopy theory of topological spaces. In addition, they provide a playground that includes other phenomena as well, such as categories. This seminar will cover the basics of this theory, focussing on the homotopy theory of spaces. Our main sources include [Gro25], [HM22], [Sch26]; parts of [Ker, Tag 00SY] may also be useful.

Each talk should be roughly 80 minutes long, accounting for questions and comments. It is up to each presenter to choose exactly what should be presented from topic, although the main theorems and definitions should always be given. Make sure to include plenty of examples in your talk, even if not explicitly asked for in the abstracts below.

**Simplicial sets and geometric realisation.** Recall the definition of the category of simplicial sets. Recall how singular homology of topological spaces factors through simplicial sets; in particular, define the homology of an arbitrary simplicial set. Define the geometric realisation $|X|$ of a simplicial set $X$. Discuss the skeletal filtration on $|X|$, in particular showing $|X|$ is a CW complex. See [HM22, Sections 2.1–2.3].

**Products of simplicial sets.** Introduce the notion of a simplicial homotopy, adding the warning that it is not an equivalence relation in general. The goal of this talk is to show that geometric realisation preserves products, from which we can deduce that it sends homotopies of topological spaces to simplicial homotopies. First deal with the case of products of standard simplices $\Delta[n] \times \Delta[k]$, decomposing it into standard simplices through the use of *shuffles*. Next, briefly introduce the notion of a convenient category of topological spaces. Show that geometric realisation $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ preserves products when working with a convenient category of topological spaces as the target. See [HM22, Section 2.5].

**Intermezzo: limits and colimits.** Cover [Tae23, Chapter 9], including at least the following. Define the notion of a limit and a colimit in a category, and discuss the specific examples of (co)products, pullbacks, and pushouts. Show that the category Set of sets has all (small) limits and colimits, and give the formula for computing these. Likewise, show that the category Ab of abelian groups has all limits and colimits. Discuss the notion of a functor preserving (co)limits. As an example, show that the forgetful functor Set $\to$ Ab preserves all limits, but not all colimits. Introduce the notion of adjoint functors. Prove that left adjoint functors preserve colimits, and that right adjoint functors preserve limits. Finally, discuss the notion of an equivalence of categories; see [Tae23, Section 6.2].

**Simplicial sets as presheaves.** Cover [HM22, Section 2.4], including at least the following. For a small category $\mathcal{C}$, introduce the presheaf category $\mathrm{PSh}(\mathcal{C})$, showing it has all limits and colimits. Define the Yoneda embedding, and prove the Yoneda lemma, and conclude that the Yoneda embedding is a fully faithful functor; see, e.g., [Gro25, Section 1.3]. Next, discuss the universal property of a presheaf category: if $\mathcal{D}$ is a category admitting all (small) colimits, then colimit-preserving functors $\mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ correspond to functors $\mathcal{C} \to \mathcal{D}$. Prove that such colimit-preserving functors $\mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ always admit a right adjoint $\mathcal{D} \to \mathrm{PSh}(\mathcal{C})$. Specialise all of this to the case of simplicial sets, using that $\mathrm{sSet} = \mathrm{PSh}(\Delta)$; in particular, show that the Yoneda lemma says that $\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], X) \cong X_n$ for all $n$. Furthermore, show how this retrieves the definition of geometric realisation, and moreover results in an adjunction

$$|-|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

between the geometric realisation and the singular complex.

**Nerves of categories.** Define the nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$, and prove it is fully faithful and admits a left adjoint; see [HM22, Section 2.4]. Describe the left adjoint $\tau$. If $\mathcal{C}$ is a small category, define its classifying space $\mathrm{B}\mathcal{C}$ as the geometric realisation $|\mathrm{N}(\mathcal{C})|$. Show that a natural transformation $F \to G$ between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ induces a simplicial homotopy between $\mathrm{N}(F)$ and $\mathrm{N}(G)$. Show that an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ induces a homotopy equivalence $\mathrm{B}\mathcal{C} \simeq \mathrm{B}\mathcal{D}$; in particular, if $\mathcal{C}$ has an initial or terminal object, then $\mathrm{B}\mathcal{C}$ is contractible. If $G$ is a (discrete) group, define $\mathrm{B}G$ as the classifying space of the category with one object $*$ whose automorphisms are given by $G$. Define the space $\mathrm{E}G$ as the classifying space of the category $\widetilde{\mathrm{E}G}$ whose objects are given by $G$, and where there is a unique morphism between every two objects. Show that $\mathrm{E}G$ is contractible and carries a continuous $G$-action, that $\mathrm{E}G/G \simeq \mathrm{B}G$, and that $\mathrm{E}G \to \mathrm{B}G$ is a universal cover. In particular, $\mathrm{B}G$ is an Eilenberg–MacLane space with fundamental group isomorphic to $G$.

**Kan complexes.** Define the notion of a Kan complex. Show that $\Delta[n]$ is never a Kan complex for $n > 0$, that $\mathrm{Sing}\, X$ is a Kan complex for every topological space $X$, and that the nerve $\mathrm{N}(\mathcal{C})$ of a category $\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid. Define the notions of a Kan fibration and a trivial Kan fibration, showing that these properties are preserved under pullbacks. Dually, discuss the notion of a saturated class for a class of morphisms ([HM22, Section 5.3]), and use this to the notion of anodyne maps of simplicial sets. Show that the saturated closure of all boundary inclusions $\{\, \partial\Delta[n] \to \Delta[n] \,\}_n$ is equal to the collection of all injections. Show that if $A \subseteq B$ and $C \subseteq D$ are inclusions one of which is anodyne, then the induced map $A \times D \cup B \times C \to B \times D$ is also anodyne. Finally, use this to conclude two things: that Kan complexes satisfy a homotopy extension property, and that simplicial homotopy is an equivalence relation for maps into a Kan complex. See [Ker, Tag 00T0 and Tag 00UG] and [HM22, Sections 5.1–5.4].

**Simplicial homotopy groups.** Cover [Ker, Tag 00V2], including at least the following. Introduce the notion of homotopy groups of pointed Kan complexes. Show that $\pi_0(X, x_0)$ can be identified with the coequaliser of $X_1 \rightrightarrows X_0$. Show that $\pi_n(X, x_0)$ carries a natural group structure for $n \geqslant 1$, which is abelian if $n \geqslant 2$. If $(X, x_0)$ is a pointed topological space, identify $\pi_n(\operatorname{Sing} X,\ x_0)$ with $\pi_n(X, x_0)$. Define the notion of a weak homotopy equivalence of simplicial sets, and show that a weak homotopy equivalence of simplicial sets is one whose geometric realisation is a weak homotopy equivalence in the usual sense. Show that the properties we expect from ordinary homotopy groups hold true in the simplicial context too; in particular, a Kan fibration leads to a long exact sequence in homotopy.

**Simplicial vs. topological I.** This is the first of two lectures to compare the homotopy theories of simplicial sets and topological spaces. The goal of this lecture is to show that for every simplicial set $X$, the unit $X \to \operatorname{Sing}|X|$ induces an isomorphism on $A$-homology for all abelian groups $A$. See [Sch26, Chapter 5].

**Simplicial vs. topological II.** Following [Sch26, Chapter 6], show that for every topological space $X$, the counit $|\operatorname{Sing} X| \to X$ is a weak homotopy equivalence. In particular, explain how this yields a natural CW approximation for arbitrary topological spaces. Finally, explain how it follows from the triangle identities that the unit $X \to \operatorname{Sing}|X|$ is a weak equivalence too; see [Sch26, Proposition 7.6].

**The homotopy category of spaces.** In the previous two talks, we showed that the unit and counit of the adjunction $|-|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top} : \operatorname{Sing}$ are weak equivalences. The goal of this talk is to deduce from this that the adjunction becomes an equivalence of categories after inverting the weak equivalences on both sides. (The resulting category is known as the *homotopy category of spaces*.) For this, introduce the notion of a (weak) localisation of a category at a class of morphisms, briefly mentioning the set-theoretic issues that can arise in general. Deduce from the previous lectures that the adjunction descends to an equivalence $\mathrm{Top}[\mathrm{w.e.}^{-1}] \simeq \mathrm{sSet}[\mathrm{w.e.}^{-1}]$. After this, show that these localisations identify with $\mathrm{Ho}(\mathrm{CW})$ and $\mathrm{Ho}(\mathrm{Kan})$, the homotopy categories of CW complexes and of Kan complexes, respectively. See [Sch26, Chapter 7].

**Kan's $\mathrm{Ex}^\infty$-functor.** Previously, we showed that for every simplicial set $X$, the unit map $X \to \operatorname{Sing}|X|$ is an anodyne map to a Kan complex, i.e., $\operatorname{Sing}|X|$ is a functorial replacement of $X$ by a Kan complex. The goal of this lecture is to give a different such replacement (Kan's $\mathrm{Ex}^\infty$-functor), which is much more efficient and does not use topological spaces in its construction. Discuss simplicial subdivision and how one defines Ex in terms of this, and how $\mathrm{Ex}^\infty$ is obtained by applying this 'infinitely many times'. Show that $\mathrm{Ex}^\infty X$ is a Kan complex for every $X$, and that the map $X \to \mathrm{Ex}^\infty X$ is anodyne, and is an isomorphism on 0-simplices. Use this construction to prove that an injective map of simplicial sets is a weak homotopy equivalence if and only if it is anodyne. See [Ker, Tag 00XF], or the original [Kan57] (note that Kan's original paper uses slightly different terminology in various places).

**Brown's finite computability.** Let $(X, x_0)$ be a finite CW complex. Brown [Bro57] showed, using simplicial methods, that for every $n$, there is a finite algorithm to determine $\pi_n(X, x_0)$ (though it is not practical for computation). To explain this algorithm, first outline Serre's method of computing homotopy groups using the Hurewicz Theorem and Eilenberg–MacLane spaces. Moving to the simplicial world, define the Kan complex $\mathrm{K}(A, n)$ for $A$ an abelian group and $n \geqslant 0$, as well as $\mathrm{E}(A, n)$. Explain the simplicial Postnikov construction, and outline how Brown's argument goes.

## References


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